

AN ESTIMATE FOR DERIVATIVE OF THE DE LA VALLÉE POUSSIN MEAN

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ABSTRACT. The de la Vallée Poussin mean for exponential weights on $(-\infty, \infty)$ was investigated in [6]. In the present paper we discuss its derivatives. An estimate for the Christoffel function plays an important role.

1. INTRODUCTION

Let $\mathbb{R} = (-\infty, \infty)$. We consider an exponential weight

$$w(x) = \exp(-Q(x))$$

on \mathbb{R} , where Q is an even and nonnegative function on \mathbb{R} . Throughout this paper we always assume that w belongs to a relevant class $\mathcal{F}(C^2+)$ (see section 2). A function $T = T_w$ defined by

$$(1.1) \quad T(x) := \frac{xQ'(x)}{Q(x)}, \quad x \neq 0$$

is very important. We call w a Freud-type weight if T is bounded, and otherwise, w is called an Erdős-type weight. For $x > 0$, the Mhaskar-Rakhmanov-Saff number (MRS number) $a_x = a_x(w)$ of $w = \exp(-Q)$ is defined by a positive root of the equation

$$(1.2) \quad x = \frac{2}{\pi} \int_0^1 \frac{a_x u Q'(a_x u)}{(1-u^2)^{1/2}} du.$$

When $w = \exp(-Q) \in \mathcal{F}(C^2+)$, Q' is positive and increasing on $(0, \infty)$, so that

$$(1.3) \quad \lim_{x \rightarrow \infty} a_x = \infty \quad \text{and} \quad \lim_{x \rightarrow +0} a_x = 0$$

and

$$(1.4) \quad \lim_{x \rightarrow \infty} \frac{a_x}{x} = 0 \quad \text{and} \quad \lim_{x \rightarrow +0} \frac{a_x}{x} = \infty$$

hold. Note that those convergences are all monotonically.

Let $\{p_n\}$ be orthogonal polynomials for a weight w , that is, p_n is the polynomial of degree n such that

$$\int_{\mathbb{R}} p_n(x) p_m(x) w(x)^2 dx = \delta_{mn}.$$

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Note that when $w(x) = \exp(-|x|^2)$, then $\{p_n\}$ are Hermite polynomials.

For $1 \leq p \leq \infty$, we denote by $L^p(I)$ the usual L^p space on an interval I in \mathbb{R} . For a function f with $fw \in L^p(\mathbb{R})$, we set

$$s_n(f)(x) := \sum_{k=0}^{n-1} b_k(f) p_k(x) \text{ where } b_k(f) = \int_{\mathbb{R}} f(t) p_k(t) w(t)^2 dt$$

for $n \in \mathbb{N}$ (the partial sum of Fourier series). The de la Vallée Poussin mean $v_n(f)$ of f is defined by

$$v_n(f)(x) := \frac{1}{n} \sum_{j=n+1}^{2n} s_j(f)(x).$$

In [6], we proved the following; Let $1 \leq p \leq \infty$ and $w \in \mathcal{F}(C^2+)$. Assume that $T(a_n) \leq Cn^{2/3-\delta}$ for some $0 < \delta \leq 2/3$ and $C > 1$. Then there exists another constant $C > 1$ such that if $fw \in L^p(\mathbb{R})$, then

$$(1.5) \quad \|v_n(f) \frac{w}{T^{1/4}}\|_{L^p(\mathbb{R})} \leq C \|fw\|_{L^p(\mathbb{R})}$$

holds for all $n \in \mathbb{N}$, and if $T^{1/4}fw \in L^p(\mathbb{R})$, then

$$(1.6) \quad \|v_n(f)w\|_{L^p(\mathbb{R})} \leq C \|T^{1/4}fw\|_{L^p(\mathbb{R})}$$

holds for all $n \in \mathbb{N}$. It is also known that

$$(1.7) \quad \left\| P' \frac{w}{T^{1/2}} \right\|_{L^p(\mathbb{R})} \leq C \left(\frac{n}{a_n} \right) \|Pw\|_{L^p(\mathbb{R})},$$

for all $P \in \mathcal{P}_n$, where \mathcal{P}_n is the set of all polynomials of degree at most n (see [5, Theorem 6.1]). Since $v_n(f) \in \mathcal{P}_{2n}$, combining (1.6) with (1.7), we have

$$(1.8) \quad \left\| v'_n(f) \frac{w}{T^{1/2}} \right\|_{L^p(\mathbb{R})} \leq C \left(\frac{n}{a_n} \right) \|T^{1/4}fw\|_{L^p(\mathbb{R})}$$

with some $C > 1$. Here we use the fact that a_n and a_{2n} are comparable (see Lemma 2.1 (1) below). The inequality (1.8) suggests us the following: if $fw \in L^p(\mathbb{R})$, then

$$(1.9) \quad \left\| v'_n(f) \frac{w}{T^{3/4}} \right\|_{L^p(\mathbb{R})} \leq C \left(\frac{n}{a_n} \right) \|fw\|_{L^p(\mathbb{R})}$$

and, if $T^{3/4}fw \in L^p(\mathbb{R})$, then

$$(1.10) \quad \|v'_n(f)w\|_{L^p(\mathbb{R})} \leq C \left(\frac{n}{a_n} \right) \|T^{3/4}fw\|_{L^p(\mathbb{R})}$$

holds?

In the present paper, we will show that (1.9) holds for all $1 \leq p \leq \infty$ and (1.10) is true for $2 \leq p \leq \infty$ at the least. More generally, as for the j th derivative $v_n^{(j)}(f)$ of $v_n(f)$, the following theorems are established.

Theorem 1.1. Let $k \geq 2$ be an integer and let $w \in \mathcal{F}_\lambda(C^4+)$ with $0 < \lambda < (k+3)/(k+2)$, and let $1 \leq p \leq \infty$. Then there exists a constant $C > 1$ such that if $1 \leq j \leq k$, and if $fw \in L^p(\mathbb{R})$, then

$$(1.11) \quad \|v_n^{(j)}(f) \frac{w}{T^{(2j+1)/4}}\|_{L^p(\mathbb{R})} \leq C \left(\frac{n}{a_n}\right)^j \|fw\|_{L^p(\mathbb{R})}$$

holds for all $n \in \mathbb{N}$.

The definition of a class $\mathcal{F}_\lambda(C^4+)$ is given in section 2.

Theorem 1.2. Let k and w be as in Theorem 1.1, and let $2 \leq p \leq \infty$. Then there exists a constant $C > 1$ such that if $1 \leq j \leq k$, and if $T^{(2j+1)/4}fw \in L^p(\mathbb{R})$, then

$$(1.12) \quad \|v_n^{(j)}(f)w\|_{L^p(\mathbb{R})} \leq C \left(\frac{n}{a_n}\right)^j \|T^{(2j+1)/4}fw\|_{L^p(\mathbb{R})}$$

holds for all $n \in \mathbb{N}$.

Theorem 1.3. Let k and w be as in Theorem 1.1, and let $1 \leq p \leq 2$. Then there exists a constant $C > 1$ such that for every $1 \leq j \leq k$ and every $T^{(2j+1)/4}fw \in L^2(\mathbb{R})$, we have

$$(1.13) \quad \|v_n^{(j)}(f)w\|_{L^p(\mathbb{R})} \leq C \left(\frac{n}{a_n}\right)^j a_n^{(2-p)/2p} \|T^{(2j+1)/4}fw\|_{L^2(\mathbb{R})}$$

for all $n \in \mathbb{N}$.

We note that when w is a Freud-type weight, then $1 \leq T \leq C$, so that,

$$(1.14) \quad \|v_n^{(j)}(f)w\|_{L^p(\mathbb{R})} \leq C \left(\frac{n}{a_n}\right)^j \|fw\|_{L^p(\mathbb{R})}$$

follows from Theorem 1.1. In [3, Chapter 3], Mhasker discussed the 1st derivative of the de la Vallée Poussin mean for Freud-type weights. Our contribution is to deal with not only Freud-type but also Erdős-type weights. In the proofs of above theorems, we use Mhasker's argument. In addition, there are two keys: one is to use mollification of exponential weights (see Lemma 2.4 below) which was obtained in [5], and another is to estimate the Christoffel functions which are done in section 3. Unfortunately, we do not know whether (1.12) holds true or not for $1 \leq p < 2$, however, we will give another estimate which holds for all $1 \leq p \leq \infty$ in section 4. A related inequality to (1.13) is also given in section 6.

Throughout this paper, we write $f(x) \sim g(x)$ for a subset $I \subset \mathbb{R}$ if there exists a constant $C \geq 1$ such that $f(x)/C \leq g(x) \leq Cf(x)$ holds for all $x \in I$. Similarly, $a_n \sim b_n$ means that $a_n/C \leq b_n \leq Ca_n$ holds for all $n \in \mathbb{N}$. We will use the same letter C to denote various positive

constants; it may vary even within a line. Roughly speaking, $C > 1$ implies that C is sufficiently large, and differently, $C > 0$ means C is a sufficiently small positive number.

2. DEFINITIONS AND LEMMAS

We say that an exponential weight $w = \exp(-Q)$ belongs to class $\mathcal{F}(C^2+)$, when $Q : \mathbb{R} \rightarrow [0, \infty)$ is a continuous and even function and satisfies the following conditions:

- (a) $Q'(x)$ is continuous in \mathbb{R} and $Q(0) = 0$.
- (b) $Q''(x)$ exists and is positive in $\mathbb{R} \setminus \{0\}$.
- (c) $\lim_{x \rightarrow \infty} Q(x) = \infty$.
- (d) The function T in (1.1) is quasi-increasing in $(0, \infty)$ (i.e. there exists $C > 1$ such that $T(x) \leq CT(y)$ whenever $0 < x < y$), and there exists $\Lambda \in \mathbb{R}$ such that

$$T(x) \geq \Lambda > 1, \quad x \in \mathbb{R} \setminus \{0\}.$$

- (e) There exists $C > 1$ such that

$$\frac{Q''(x)}{|Q'(x)|} \leq C \frac{|Q'(x)|}{Q(x)}, \quad \text{a.e. } x \in \mathbb{R}.$$

Moreover, if there also exist a compact subinterval $J(\ni 0)$ of \mathbb{R} , and $C > 1$ such that

$$C \frac{Q''(x)}{|Q'(x)|} \geq \frac{|Q'(x)|}{Q(x)}, \quad \text{a.e. } x \in \mathbb{R} \setminus J.$$

Let $\lambda > 0$. We write $w \in \mathcal{F}_\lambda(C^2+)$ if there exist $K > 1$ and $C > 1$ such that for all $|x| \geq K$,

$$(2.1) \quad \frac{|Q'(x)|}{Q(x)^\lambda} \leq C$$

holds. We also write $w \in \mathcal{F}_\lambda(C^3+)$, if $Q \in C^3(\mathbb{R} \setminus \{0\})$ and

$$\left| \frac{Q^{(3)}(x)}{Q''(x)} \right| \leq C \left| \frac{Q''(x)}{Q'(x)} \right| \quad \text{and} \quad \frac{|Q'(x)|}{Q(x)^\lambda} \leq C$$

holds for every $|x| \geq K$. Moreover, we write $w \in \mathcal{F}_\lambda(C^4+)$, if $Q \in C^4(\mathbb{R} \setminus \{0\})$ and

$$\left| \frac{Q^{(3)}(x)}{Q''(x)} \right| \sim \left| \frac{Q''(x)}{Q'(x)} \right|, \quad \left| \frac{Q^{(4)}(x)}{Q^{(3)}(x)} \right| \leq C \left| \frac{Q''(x)}{Q'(x)} \right| \quad \text{and} \quad \frac{|Q'(x)|}{Q(x)^\lambda} \leq C$$

holds for every $|x| \geq K$. Clearly $\mathcal{F}_\lambda(C^4+) \subset \mathcal{F}_\lambda(C^3+) \subset \mathcal{F}_\lambda(C^2+) \subset \mathcal{F}(C^2+)$.

A typical example of Freud-type weight is $w = \exp(-|x|^\alpha)$ with $\alpha > 1$. It belongs to $\mathcal{F}_\lambda(C^4+)$ for $\lambda = 1$. For $u \geq 0$, $\alpha > 0$ with $\alpha + u > 1$ and $l \in \mathbb{N}$, we set

$$Q(x) := |x|^u (\exp_l(|x|^\alpha) - \exp_l(0)),$$

where $\exp_l(x) := \exp(\exp(\exp(\cdots(\exp x))))$ (l -times). Then $w = \exp(-Q(x))$ is an Erdős-type weight, which belongs to $\mathcal{F}_\lambda(C^4+)$ for $\lambda > 1$ (see[1]).

In the following lemmas we fix $w \in \mathcal{F}(C^2+)$.

Lemma 2.1. Fix $L > 0$. Then we have

- (1) $a_t \sim a_{Lt}$ on $t > 0$ (see [2, Lemma 3.5 (a)]).
- (2) $Q(a_t) \sim Q(a_{Lt})$, $Q'(a_t) \sim Q'(a_{Lt})$ and $T(a_{Lt}) \sim T(a_t)$ on $t > 0$ (see [2, Lemma 3.5 (b)]).

- (3) $\frac{1}{T(a_t)} \sim \left|1 - \frac{a_{Lt}}{a_t}\right|$ on $t > 0$ (see [2, Lemma 3.11 (3.52)]).

- (4) $\frac{t}{\sqrt{T(a_t)}} \sim Q(a_t)$ and $\frac{t\sqrt{T(a_t)}}{a_t} \sim |Q'(a_t)|$ on $t > 0$ (see [2, Lemma 3.4 (3.18) and (3.17)]).

- (5) Assume that w is an Erdős-type weight. Then for every $\eta > 0$, there exists a constant $C_\eta > 1$ such that

$$(2.2) \quad a_x \leq C_\eta x^\eta \quad (x \geq 1)$$

(see [4, Proposition 3 (3.8)]).

Lemma 2.2. ([2, Theorem 1.9 (a)]) Let $1 \leq p \leq \infty$. Then

$$(2.3) \quad \|Pw\|_{L^p(\mathbb{R})} \leq 2\|Pw\|_{L^p([-a_n, a_n])}$$

for every $n \in \mathbb{N}$ and every $P \in \mathcal{P}_n$.

Lemma 2.3. (1) There exist constants $C_1 > 1$ and $c_0 > 0$ such that if $|x - t| < c_0/T(x)$ then $T(t)/C_1 \leq T(x) \leq C_1 T(t)$ holds (cf. [2, Theorem 3.2 (e)] see also [6, Lemma 3.4]).

(2) There exist a constant $C_2 > 1$ such that for any $n \in \mathbb{N}$, if $|t|, |x| < a_{2n}$ and $|x - t| \leq a_n/n$ then $T(t)/C_2 \leq T(x) \leq C_2 T(t)$ holds (see [6, (4.6)]).

Lemma 2.4. ([5, Theorems 4.1 and 4.2]) Let $m = 1, 2$ and let $w \in \mathcal{F}_\lambda(C^{2+m}+)$ with $0 < \lambda < (m+2)/(m+1)$. For every $\alpha \in \mathbb{R}$, we can construct a new weight $w^* \in \mathcal{F}_\lambda(C^{1+m}+)$ such that

$$(2.4) \quad w^*(x) \sim T(x)^\alpha w(x) \text{ and } T^*(x) \sim T(x)$$

on \mathbb{R} , and

$$(2.5) \quad a_{x/c} \leq a_x^* \leq a_{cx}$$

holds on \mathbb{R} with some constant $c > 1$, where T^* and a_x^* are corresponding ones defined in (1.1) and (1.2) with respect to w^* respectively.

Using the above lemma, we obtain the following assertions. First one is a generalization of (1.7). Second assertion was shown in [5, Corollary 6.2] under some additional assumption.

Lemma 2.5. Let $w \in \mathcal{F}_\lambda(C^3+)$ with $0 < \lambda < 3/2$ and let $1 \leq p \leq \infty$. For $j \in \mathbb{N}$, there exists a constant $C_3 > 1$ such that for every $n \in \mathbb{N}$ and every $P \in \mathcal{P}_n$, we have

$$(2.6) \quad \left\| P^{(j)} \frac{w}{T^{j/2}} \right\|_{L^p(\mathbb{R})} \leq C_3 \left(\frac{n}{a_n} \right)^j \|Pw\|_{L^p(\mathbb{R})}$$

and if we further assume that $w \in \mathcal{F}_\lambda(C^4+)$ with $0 < \lambda < 4/3$, then there exists a constant $C_4 > 1$ such that

$$(2.7) \quad \|P^{(j)}w\|_{L^p(\mathbb{R})} \leq C_4 \left(\frac{n}{a_n} \right)^j \|T^{j/2}Pw\|_{L^p(\mathbb{R})}$$

also holds.

Proof. For $i = 1, \dots, j$, let $w_i^* \in \mathcal{F}_\lambda(C^2+)$ be a weight obtained in Lemma 2.4 for $\alpha = -(i-1)/2$. Then, since $P^{(j)} \in \mathcal{P}_{n-j}$, by (1.7) for w_j^* and by (2.4) and (2.5), there exists a constant $C > 1$ such that

$$\left\| P^{(j)} \frac{w_j^*}{T^{1/2}} \right\|_{L^p(\mathbb{R})} \leq C \left(\frac{n-j+1}{a_{(n-j+1)/c}} \right) \|P^{(j-1)}w_j^*\|_{L^p(\mathbb{R})}.$$

Since $w_j^*(x) \sim T(x)^{-1/2}w_{j-1}^*(x)$, we also see

$$\left\| P^{(j)} \frac{w_j^*}{T^{1/2}} \right\|_{L^p(\mathbb{R})} \leq C \left(\frac{n-j+1}{a_{(n-j+1)/c}} \right) \left\| P^{(j-1)} \frac{w_{j-1}^*}{T^{1/2}} \right\|_{L^p(\mathbb{R})}.$$

Repeating this process, we have

$$\begin{aligned} \left\| P^{(j)} \frac{w}{T^{j/2}} \right\|_{L^p(\mathbb{R})} &\leq C \left\| P^{(j)} \frac{w_j^*}{T^{1/2}} \right\|_{L^p(\mathbb{R})} \\ &\leq C^{j+1} \left(\frac{n-j+1}{a_{(n-j+1)/c}} \right) \cdots \left(\frac{n}{a_{n/c}} \right) \|Pw\|_{L^p(\mathbb{R})} \\ &\leq C_3 \left(\frac{n}{a_n} \right)^j \|Pw\|_{L^p(\mathbb{R})}, \end{aligned}$$

where we use Lemma 2.1 (1).

For (2.7), we first remark that if $w \in \mathcal{F}(C^3+)$, then

$$(2.8) \quad \|P'w\|_{L^p(\mathbb{R})} \leq C_4 \left(\frac{n}{a_n} \right) \|T^{1/2}Pw\|_{L^p(\mathbb{R})}$$

holds true (see [5, Theorem 1.1]). This is the case $j = 1$. To show general case $j > 1$, we consider a weight $w_i^{**} \in \mathcal{F}_\lambda(C^3+)$ in Lemma 2.4 for $w \in \mathcal{F}_\lambda(C^4+)$ with $\alpha = (i-1)/2$ ($i = 1, \dots, j$). Applying $P^{(j-i)}$ and w_i^{**} to (2.8) and repeating this process for $i = 1, \dots, j$, we obtain (2.7) as in (2.6). This completes the proof.

Lemma 2.6. Let $k \in \mathbb{N} \cup \{0\}$ and $w \in \mathcal{F}_\lambda(C^2+)$ with $0 < \lambda < (k+2)/(k+1)$. Then there exist constants $C_5 > 1$ and $\delta > 0$ such that

$$(2.9) \quad T(a_n) \leq C_5 n^{2/(2k+3)-\delta}$$

holds for all $n \in \mathbb{N}$.

Proof. We may assume that $w = \exp(-Q)$ is an Erdős-type weight. By (2.1), $|Q'(x)|/Q(x)^\lambda \leq C$ with some constant $C > 1$. Hence Lemma 2.1 (4) gives us

$$\frac{n\sqrt{T(a_n)}}{a_n} \left(\frac{n}{\sqrt{T(a_n)}} \right)^{-\lambda} \leq C,$$

that is, $T(a_n) \leq C a_n^{2/(\lambda+1)} n^{2(\lambda-1)/(\lambda+1)}$. Since $\lambda < (k+2)/(k+1)$, we can choose $\delta > 0$ and $\eta > 0$ such that $2(\lambda-1)/(\lambda+1) + \delta + 2\eta < 2/(2k+3)$. Hence (2.9) follows from Lemma 2.1 (5). This completes the proof.

Lemma 2.7. Let $w \in \mathcal{F}_\lambda(C^2+)$ with $0 < \lambda < 2$. Then there exists a constant $C_6 > 1$ such that for every $n \in \mathbb{N}$, if $|t|, |x| < a_{2n}$ and if $|t - x| < a_n/(n\sqrt{T(x)})$ then

$$(2.10) \quad w(t)/C_6 \leq w(x) \leq C_6 w(t)$$

Proof. By Lemma 2.3 (2), we have $T(t)/C_2 \leq T(x) \leq C_2 T(t)$, and by (1.3) we can write $|t| = a_s$. Then $a_s \leq a_{2n}$ implies $s \leq 2n$. Hence (1.4) and Lemma 2.1(1) show $s a_n/(n a_s) \leq C_7$ with some constant $C_7 > 1$. Since $|Q'(t)| \leq C s \sqrt{T(a_s)}/a_s$ by Lemma 2.1 (4), we have

$$\begin{aligned} |Q'(t)||t - x| &\leq C \frac{s \sqrt{T(a_s)}}{a_s} \frac{a_n}{n} \frac{1}{\sqrt{T(x)}} \\ &\leq C \frac{a_n}{n} \frac{s}{a_s} \frac{\sqrt{T(t)}}{\sqrt{T(x)}} \leq C C_7 \sqrt{C_2}. \end{aligned}$$

Similarly, we see $|Q'(x)||t - x| \leq C C_7$. Hence if we put $C_6 = e^{C C_7 \sqrt{C_2}}$, then $|Q'(t)||t - x| \leq \log C_6$ and $|Q'(x)||t - x| < \log C_6$ hold true. From mean value theorem for differential calculus, there exists θ between x and t such that

$$\frac{w(x)}{w(t)} = \exp(Q(t) - Q(x)) = \exp(Q'(\theta)(t - x)).$$

Since Q' is increasing, $|Q'(\theta)(x - t)| \leq \max\{|Q'(x)|, |Q'(t)|\}|x - t| \leq \log C_6$, which shows (2.10) immediately. This completes the proof.

3. ESTIMATES FOR CHRISTOFFEL FUNCTIONS

By definition, the partial sum of Fourier series is given by

$$(3.1) \quad s_n(f)(x) = \int_{\mathbb{R}} K_n(x, t) f(t) w(t)^2 dt,$$

where

$$(3.2) \quad K_n(x, t) = \sum_{k=0}^{n-1} p_k(x) p_k(t).$$

It is known that by the Cristoffel-Darboux formula

$$(3.3) \quad K_n(x, t) = \frac{\gamma_{n-1}}{\gamma_n} \frac{p_n(x)p_{n-1}(t) - p_n(t)p_{n-1}(x)}{x - t}$$

holds, where γ_n and γ_{n-1} are the leading coefficients of p_n and p_{n-1} , respectively. Then

$$(3.4) \quad a_n \sim \frac{\gamma_{n-1}}{\gamma_n}$$

also holds (see [2, Lemma 13.9]).

The Christoffel function $\lambda_n(x) = \lambda_n(w, x)$ is defined by

$$\lambda_n(x) := \frac{1}{K_n(x, x)} = \left(\sum_{k=0}^{n-1} p_k(x)^2 \right)^{-1}.$$

Then

$$(3.5) \quad \lambda_n(x) = \inf_{P \in \mathcal{P}_{n-1}} \frac{1}{P(x)^2} \int_{\mathbb{R}} |P(t)w(t)|^2 dt.$$

holds on \mathbb{R} . We use derivative versions of (3.5). The following equality is also established.

Proposition 3.1. Let $0 \leq j < n$. Then for every $x \in \mathbb{R}$, we have

$$(3.6) \quad \left(\sum_{k=0}^{n-1} (p_k^{(j)}(x))^2 \right)^{-1} = \inf_{P \in \mathcal{P}_{n-1}} \frac{1}{(P^{(j)}(x))^2} \int_{\mathbb{R}} |P(t)w(t)|^2 dt.$$

Proof. In [3, Theorem 1.3.2], we see

$$\left(\sum_{k=0}^{n-1} \Phi(p_k)^2 \right)^{-1} = \inf_{P \in \mathcal{P}_{n-1}} \frac{1}{(\Phi(P))^2} \int_{\mathbb{R}} |P(t)w(t)|^2 dt$$

for any linear functional Φ on polynomials. (3.6) follows if we consider $\Phi(P) = P^{(j)}(x)$.

The following estimate plays an important role in our later argument. We use C_m ($m = 1, \dots, 6$), which are constants in lemmas of the previous section.

Proposition 3.2. Let $k \geq 2$ be an integer and let $w \in \mathcal{F}_\lambda(C^4+)$ with $0 < \lambda < (k+3)/(k+2)$. Then there exists a constant $C_8 > 1$ such that for every $1 \leq j \leq k$ and every $n \in \mathbb{N}$,

$$(3.7) \quad \frac{w(x)^2}{T(x)^{(2j+1)/2}} \sum_{k=0}^{n-1} (p_k^{(j)}(x))^2 \leq C_8 \left(\frac{n}{a_n} \right)^{2j+1}.$$

Proof. It is enough to show (3.7) for sufficiently large n . By Proposition 3.1, (3.7) follows from

$$(3.8) \quad \left(\frac{a_n}{n}\right)^{2j+1} \frac{w(x)^2}{T(x)^{(2j+1)/2}} \leq C_8 \frac{1}{(P^{(j)}(x))^2} \int_{\mathbb{R}} |P(t)w(t)|^2 dt$$

for all $P \in \mathcal{P}_{n-1}$. Now take $P \in \mathcal{P}_{n-1}$ be arbitrarily. By Lemma 2.2, we can choose $\zeta \in \mathbb{R}$ such that $|\zeta| \leq a_{n-1}$ and satisfies

$$(3.9) \quad \|wP\|_{L^\infty(\mathbb{R})} \leq 2|w(\zeta)P(\zeta)|.$$

Let $0 < c_1 \leq 1$. Lemma 2.6 gives us $T(a_n) \leq C_5 n^{1-\delta'}$ with some $\delta' > 0$, so that if $t \in \mathbb{R}$ satisfies

$$(3.10) \quad |t - \zeta| \leq c_1 \frac{a_n}{n} \frac{1}{\sqrt{T(\zeta)}},$$

then

$$|t| \leq |\zeta| + |\zeta - t| \leq |\zeta| + c_1 \frac{a_n}{n} \frac{1}{\sqrt{T(\zeta)}} \leq a_{n-1} + \frac{a_n}{n} \leq a_n + \frac{C_5}{n^{\delta'}} \frac{a_n}{T(a_n)}.$$

Since there exists a constant $C > 1$ such that $a_n + a_n/(CT(a_n)) \leq a_{2n}$ by Lemma 2.1 (3), if we take $n_0 \in \mathbb{N}$ such that $n_0^{\delta'} > CC_5$, then

$$(3.11) \quad |t| \leq a_{2n}$$

for all $n \geq n_0$. Hence by Lemma 2.7, $w(t)/C_6 \leq w(\zeta) \leq C_6 w(t)$ holds. By monotonicity of w , $w(u)/C_6 \leq w(\zeta) \leq C_6 w(u)$ also holds for every u between t and ζ . Moreover, since T is quasi-increasing, Lemma 2.3 (2) shows $\sqrt{T(u)} \leq C\sqrt{T(\zeta)}$ with some $C > 1$. Then using (2.6) for $p = \infty$ and $j = 0$, we have

$$\begin{aligned} |P(\zeta)| - |P(t)| &\leq |P(t) - P(\zeta)| = \left| \int_{\zeta}^t P'(u) du \right| \\ &\leq CC_6 \frac{\sqrt{T(\zeta)}}{w(\zeta)} \left| \int_{\zeta}^t \frac{1}{\sqrt{T(u)}} w(u) P'(u) du \right| \\ &\leq CC_6 |t - \zeta| \frac{\sqrt{T(\zeta)}}{w(\zeta)} \left\| \frac{w}{\sqrt{T}} P' \right\|_{L^\infty(\mathbb{R})} \\ &\leq CC_6 C_3 |t - \zeta| \frac{\sqrt{T(\zeta)}}{w(\zeta)} \frac{n}{a_n} \|wP\|_{L^\infty(\mathbb{R})} \\ &\leq 2c_1 CC_6 C_3 |P(\zeta)| \end{aligned}$$

by (3.9) and (3.10). Consequently, if we take $c_1 > 0$ so small that $2c_1 CC_6 C_3 < 1/2$, we have

$$(3.12) \quad |P(t)| \geq \frac{1}{2} |P(\zeta)| \quad \text{if } |t - \zeta| \leq c_1 \frac{a_n}{n} \frac{1}{\sqrt{T(\zeta)}}.$$

Since $C_2 T(t) \geq T(\zeta)$ and $C_6 w(t) \geq w(\zeta)$, (3.9) and (3.12) show

$$\begin{aligned}
\int_{\mathbb{R}} \sqrt{T(t)} |P(t)|^2 w(t)^2 dt &\geq \frac{\sqrt{T(\zeta)}}{\sqrt{C_2}} \int_{|t-\zeta| \leq c_1(a_n/(n\sqrt{T(\zeta)}))} |P(t)|^2 w(t)^2 dt \\
&\geq \frac{\sqrt{T(\zeta)}}{\sqrt{C_2}} \frac{|P(\zeta)|^2 w(\zeta)^2}{4} \frac{a_n}{C_6^2} c_1 \frac{1}{n \sqrt{T(\zeta)}} \\
&\geq \frac{c_1}{4\sqrt{C_2}} \frac{1}{C_6^2} \frac{a_n}{n} \frac{\|wP\|_{L^\infty(\mathbb{R})}^2}{4} \\
&=: \frac{1}{C_0} \frac{a_n}{n} \|wP\|_{L^\infty(\mathbb{R})}^2.
\end{aligned}$$

We note that in the above argument we only use the fact that $w \in \mathcal{F}_\lambda(C^3+)$. If $w \in \mathcal{F}_\lambda(C^4+)$, we can construct $w^* \in \mathcal{F}_\lambda(C^3+)$ such that $w^*(x) \sim T(x)^{-1/4} w(x)$ by Lemma 2.4. Then it follows from (2.6) for $p = \infty$ that for every $x \in \mathbb{R}$,

$$\begin{aligned}
\int_{\mathbb{R}} \sqrt{T^*(t)} |P(t)|^2 w^*(t)^2 dt &\geq \frac{1}{C_0} \frac{a_n^*}{n} \|w^* P\|_{L^\infty(\mathbb{R})}^2 \\
&\geq \frac{1}{C_0 C_3} \frac{a_n^*}{n} \left(\frac{a_{n-1}^*}{n-1} \right)^{2j} \left\| \frac{w^*}{(T^*)^{j/2}} P^{(j)} \right\|_{L^\infty(\mathbb{R})}^2 \\
&\geq \frac{1}{C_0 C_3} \left(\frac{a_n^*}{n} \right)^{2j+1} \frac{w^*(x)^2}{T^*(x)^j} |P^{(j)}(x)|^2,
\end{aligned}$$

and hence by (2.4) and (2.5) we see

$$\begin{aligned}
\int_{\mathbb{R}} |P(t)|^2 w^2(t) dt &\geq \frac{1}{C} \int_{\mathbb{R}} \sqrt{T^*(t)} |P(t)|^2 w^*(t)^2 dt \\
&\geq \frac{1}{C C_0 C_3} \left(\frac{a_n^*}{n} \right)^{2j+1} \frac{w^*(x)^2}{T^*(x)^j} |P^{(j)}(x)|^2 \\
&\geq \frac{1}{C} \left(\frac{a_{n/c}}{n} \right)^{2j+1} \frac{w(x)^2}{T(x)^{(2j+1)/2}} |P^{(j)}(x)|^2.
\end{aligned}$$

This together with Lemma 2.1 (1) shows (3.8) and the proof is completed.

4. PROOF OF THEOREM 1.1

In the remaining sections, we again use C_m ($m = 1, \dots, 6$) without notice, which are constants in Lemmas of the previous section.

Let $1 \leq p \leq \infty$, $k \geq 2$, $w \in \mathcal{F}_\lambda(C^4+)$ with $0 < \lambda < (k+3)/(k+2)$ and let $1 \leq j \leq k$. Due to Lemma 2.4, there is $w^* \in \mathcal{F}_\lambda(C^3+)$ such that $w^*(x) \sim T(x)^{-(2j+1)/4} w(x)$. Take $fw \in L^p(\mathbb{R})$ arbitrarily. Since

$v_n^{(j)}(f) \in \mathcal{P}_{2n-j}$, applying w^* to (2.7), we have

$$\begin{aligned}
& \left\| v_n^{(j)}(f) \frac{w}{T^{(2j+1)/4}} \right\|_{L^p(\mathbb{R})} \leq C \|v_n^{(j)}(f) w^*\|_{L^p(\mathbb{R})} \\
& \leq C \left(\frac{2n-j}{a_{2n-j}^*} \right)^j \|(T^*)^{j/2} v_n(f) w^*\|_{L^p(\mathbb{R})} \\
& \leq C \left(\frac{n}{a_{2n/c}} \right)^j \left\| v_n(f) \frac{w}{T^{1/4}} \right\|_{L^p(\mathbb{R})} \\
& \leq C \left(\frac{n}{a_n} \right)^j \|fw\|_{L^p(\mathbb{R})}.
\end{aligned}$$

Here we use Lemma 2.1 (1), (2.4) and (2.5). The last inequality follows from (1.5). This completes the proof of Theorem 1.1.

By a similar argument as above, we also have

$$(4.1) \quad \|v_n^{(j)}(f)w\|_{L^p(\mathbb{R})} \leq C \left(\frac{n}{a_n} \right)^j T(a_n)^{(2j+1)/4} \|fw\|_{L^p(\mathbb{R})}.$$

In fact, take $w^* \in \mathcal{F}_\lambda(C^3+)$ such that $w^*(x) \sim T^{j/2}(x)w(x)$. Then by (2.7) for w and by Lemma 2.4 for w^* , we have

$$\begin{aligned}
& \|v_n^{(j)}(f)w\|_{L^p(\mathbb{R})} \leq C \left(\frac{n}{a_n} \right)^j \|T^{j/2} v_n(f)w\|_{L^p(\mathbb{R})} \\
& \leq C \left(\frac{n}{a_n} \right)^j \|v_n(f)w^*\|_{L^p([-a_{2n}^*, a_{2n}^*])} \\
& \leq C \left(\frac{n}{a_n} \right)^j \left\| v_n(f) T^{(2j+1)/4} \frac{w}{T^{1/4}} \right\|_{L^p([-a_{2cn}, a_{2cn}])} \\
& \leq C \left(\frac{n}{a_n} \right)^j T(a_n)^{(2j+1)/4} \left\| v_n(f) \frac{w}{T^{1/4}} \right\|_{L^p([-a_{2cn}, a_{2cn}])} \\
& \leq C \left(\frac{n}{a_n} \right)^j T(a_n)^{(2j+1)/4} \|fw\|_{L^p(\mathbb{R})}.
\end{aligned}$$

Note that by Lemma 2.1 (2), $T(x) \leq CT(a_{2cn}) \leq CT(a_n)$ holds for all $x \in [-a_{2cn}, a_{2cn}]$, because T is quasi-increasing.

5. PROOF OF THEOREM 1.2

Let $k \geq 2$, $w \in \mathcal{F}_\lambda(C^4+)$ with $0 < \lambda < (k+3)/(k+2)$ and let $1 \leq j \leq k$. We first show (1.12) for the case $p = \infty$. Suppose that $T^{(2j+1)/4}fw \in L^\infty(\mathbb{R})$. Since $v_n^{(j)}(f) \in \mathcal{P}_{2n}$, by Lemma 2.2, it is sufficient to show

$$(5.1) \quad |v_n^{(j)}(f)(x)w(x)| \leq C \left(\frac{n}{a_n} \right)^j \|T^{(2j+1)/4}fw\|_{L^\infty(\mathbb{R})}$$

for every $|x| \leq a_{2n}$. Now we set

$$A_n := \{t \in \mathbb{R}; |t - x| < \frac{a_{2n}}{2n}\}, \quad B_n := \{t \in \mathbb{R}; \frac{a_{2n}}{2n} \leq |t - x| < \frac{c_0}{T(x)}\}$$

and $C_n := \mathbb{R} \setminus (A_n \cup B_n)$, where $c_0 > 0$ is a constant in Lemma 2.3 (1). Then as in the proof of (3.11), there exists $n_0 \in \mathbb{N}$ such that if $n \geq n_0$ and $t \in A_n$, then $|t| \leq a_{4n}$ holds. Hence Lemma 2.3 (2) implies

$$(5.2) \quad T(t)/C_2 \leq T(x) \leq C_2 T(t) \quad (t \in A_n).$$

Since T is bounded on $[-a_{4n_0}, a_{4n_0}]$, we may assume that (5.2) holds for all $n \in \mathbb{N}$. Also by Lemma 2.3 (1),

$$(5.3) \quad T(t)/C_1 \leq T(x) \leq C_1 T(t) \quad (t \in B_n)$$

holds true. Let $g(t) := f(t)\chi_{A_n}(t)$, where χ_A is the characteristic function of a set A and put $h(t) = f(t) - g(t)$. Since

$$\int_{\mathbb{R}} \left(\sum_{k=0}^{m-1} p_k^{(j)}(x) p_k(t) \right)^2 w(t)^2 dt = \sum_{k=0}^{m-1} (p_k^{(j)}(x))^2,$$

(3.2), (5.2) and the Schwarz inequality show that

$$\begin{aligned} & |s_m^{(j)}(g)(x)w(x)| \\ & \leq w(x) \int_{\mathbb{R}} \left| g(t) \sum_{k=0}^{m-1} p_k^{(j)}(x) p_k(t) w(t)^2 \right| dt \\ & \leq \left(\sum_{k=0}^{m-1} (p_k^{(j)}(x))^2 w(x)^2 \right)^{1/2} \left(\int_{A_n} |f(t)w(t)|^2 dt \right)^{1/2} \\ & \leq C_2^{(2j+1)/4} \left(\sum_{k=0}^{m-1} \frac{w(x)^2}{T(x)^{(2j+1)/2}} (p_k^{(j)}(x))^2 \right)^{1/2} \left(\int_{A_n} |T(t)^{(2j+1)/4} f(t)w(t)|^2 dt \right)^{1/2} \\ & \leq C \left(\sum_{k=0}^{m-1} \frac{w(x)^2}{T(x)^{(2j+1)/2}} (p_k^{(j)}(x))^2 \right)^{1/2} \|T^{(2j+1)/4} f w\|_{L^\infty(\mathbb{R})} \left(\frac{a_{2n}}{2n} \right)^{1/2}. \end{aligned}$$

Since $v_n^{(j)}(g)(x) = (1/n) \sum_{m=n+1}^{2n} s_m^{(j)}(g)(x)$, Proposition 3.2 gives us

$$(5.4) \quad |v_n^{(j)}(g)(x)w(x)| \leq C \left(\frac{n}{a_n} \right)^j \|T^{(2j+1)/4} f w\|_{L^\infty(\mathbb{R})}$$

for all $x \in R$ with $|x| \leq a_{2n}$.

To estimate $v_n^{(j)}(h)$, we use (3.3). For $i = 0, 1, \dots, j$, we put

$$\begin{aligned} & v_{n,i}(h)(x) \\ &:= \frac{1}{n} \sum_{m=n+1}^{2n} \frac{\gamma_{m-1}}{\gamma_m} \int_{\mathbb{R}} h(t) \frac{p_m^{(j-i)}(x)p_{m-1}(t) - p_{m-1}^{(j-i)}(x)p_m(t)}{(x-t)^{i+1}} w(t)^2 dt \\ &= \frac{1}{n} \sum_{m=n+1}^{2n} \frac{\gamma_{m-1}}{\gamma_m} (b_{m-1}(h_i)p_m^{(j-i)}(x) - b_m(h_i)p_{m-1}^{(j-i)}(x)), \end{aligned}$$

where

$$h_i(t) := \frac{h(t)}{(x-t)^{i+1}} \quad \text{and} \quad b_k(h_i) := \int_{\mathbb{R}} h_i(t)p_k(t)w(t)^2 dt \quad (k \in \mathbb{N} \cup \{0\}).$$

Then

$$(5.5) \quad v_n^{(j)}(h)(x) = \sum_{i=0}^j (-1)^i \binom{j}{i} v_{n,i}(h)(x).$$

By (3.4), the Schwarz inequality and Proposition 3.2, we have

$$\begin{aligned} & |v_{n,i}(h)(x)w(x)| \\ &\leq \frac{1}{n} \sum_{m=0}^{2n} \left| \frac{\gamma_{m-1}}{\gamma_m} 2p_m^{(j-i)}(x)b_m(h_i)w(x) \right| \\ &\leq C \frac{a_n}{n} \left(w(x)^2 \sum_{m=0}^{2n} (p_m^{(j-i)}(x))^2 \right)^{1/2} \left(\sum_{m=0}^{2n} |b_m(h_i)|^2 \right)^{1/2} \\ &\leq C \frac{a_n}{n} \left(\frac{w(x)^2}{T(x)^{(2(j-i)+1)/2}} \sum_{m=0}^{2n} (p_m^{(j-i)}(x))^2 \right)^{1/2} \left(T(x)^{(2(j-i)+1)/2} \sum_{m=0}^{2n} |b_m(h_i)|^2 \right)^{1/2} \\ &\leq C \left(\frac{n}{a_n} \right)^{(2(j-i)-1)/2} \left(T(x)^{(2(j-i)+1)/2} \sum_{m=0}^{2n} |b_m(h_i)|^2 \right)^{1/2}. \end{aligned}$$

The Bessel inequality implies that

$$\sum_{m=0}^{2n} |b_m(h_i)|^2 \leq \int_{\mathbb{R}} \left| \frac{h(t)}{(x-t)^{i+1}} \right|^2 w(t)^2 dt = \int_{B_n \cup C_n} \frac{|f(t)w(t)|^2}{(x-t)^{2(i+1)}} dt$$

and hence, by (5.3), we have

$$\begin{aligned} & T(x)^{(2(j-i)+1)/2} \int_{B_n} \frac{|f(t)w(t)|^2}{(x-t)^{2(i+1)}} dt \\ &\leq C_1^{(2(j-i)+1)/2} \int_{B_n} \frac{|T(t)^{(2(j-i)+1)/4} f(t)w(t)|^2}{(x-t)^{2(i+1)}} dt \\ &\leq C \|T^{(2(j-i)+1)/4} f w\|_{L^\infty(\mathbb{R})}^2 \int_{|x-t| > \frac{a_{2n}}{2n}} \frac{1}{(x-t)^{2(i+1)}} dt \end{aligned}$$

$$\begin{aligned}
&\leq C \|T^{(2(j-i)+1)/4} fw\|_{L^\infty(\mathbb{R})}^2 \left(\frac{n}{a_n}\right)^{2i+1} \\
&\leq C \|T^{(2j+1)/4} fw\|_{L^\infty(\mathbb{R})}^2 \left(\frac{n}{a_n}\right)^{2i+1},
\end{aligned}$$

because $T \geq 1$. On the other hand, if $|x| \leq a_{2n}$ then $T(x) \leq CT(a_n)$, so that

$$\begin{aligned}
&T(x)^{(2(j-i)+1)/2} \int_{C_n} \frac{|f(t)w(t)|^2}{(x-t)^{2(i+1)}} dt \\
&\leq C \|fw\|_{L^\infty(\mathbb{R})}^2 T(x)^{(2(j-i)+1)/2} \int_{\frac{c_0}{T(x)} \leq |x-t|} \frac{1}{(x-t)^{2(i+1)}} dt \\
&\leq C \|fw\|_{L^\infty(\mathbb{R})}^2 T(x)^{(2(i+j)+3)/2} \\
&\leq C \|T^{(2j+1)/4} fw\|_{L^\infty(\mathbb{R})}^2 T(a_n)^{(2(i+k)+3)/2}.
\end{aligned}$$

Moreover

$$(5.6) \quad T(a_n)^{(2(i+k)+3)/2} \leq C \left(\frac{n}{a_n}\right)^{2i+1}$$

holds. In fact, to show this we may assume that w is an Erdős-type weight. Then by Lemma 2.1 (5) and Lemma 2.6, we have

$$T^{(2k+3)/2}(a_n) \leq C n^{(2/(2k+3)-\delta)((2k+3)/2)} \leq C n^{1-\delta'} \leq C \left(\frac{n}{a_n}\right).$$

Similarly

$$\begin{aligned}
T^{(2(i+k)+3)/2}(x) &\leq CT(a_n)^{(4k+3)/2} \leq C n^{(2/(2k+3)-\delta)((4k+3)/2)} \\
&\leq C n^{2-\delta''} \leq C \left(\frac{n}{a_n}\right)^2 \leq C \left(\frac{n}{a_n}\right)^{2i+1}
\end{aligned}$$

holds for $i \geq 1$. Combining the above estimates, we thus have

$$\begin{aligned}
|v_{n,i}(h)(x)w(x)| &\leq C \left(\frac{n}{a_n}\right)^{(2(j-i)-1)/2} \left(T(x)^{(2(j-i)+1)/2} \sum_{m=0}^{2n} |b_m(h_i)|^2\right)^{1/2} \\
&\leq C \left(\frac{n}{a_n}\right)^{(2(j-i)-1)/2} \|T^{(2j+1)/4} fw\|_{L^\infty(\mathbb{R})} \left(\frac{n}{a_n}\right)^{(2i+1)/2} \\
&\leq C \left(\frac{n}{a_n}\right)^j \|T^{(2j+1)/4} fw\|_{L^\infty(\mathbb{R})}.
\end{aligned}$$

It follows from (5.5) that

$$|v_n^{(j)}(h)(x)w(x)| \leq C \left(\frac{n}{a_n}\right)^j \|T^{(2j+1)/4} fw\|_{L^\infty(\mathbb{R})}.$$

This together with (5.4) shows (5.1).

We will prove (1.12) for $p = 2$ in the next section. Then using the Riesz-Thorin interpolation theorem for an operator

$$F : f \mapsto wv_n^{(j)}\left(\frac{f}{w}\right),$$

we obtain (1.12) for all $2 \leq p \leq \infty$. This completes the proof of Theorem 1.2.

6. PROOF OF THEOREM 1.3

Let $1 \leq p \leq 2$ and $T^{(2j+1)}fw \in L^2(\mathbb{R})$. We use the same notations as in the previous section. Then as in the estimate of $s_m^{(j)}(g)$ in the previous section, we have

$$(6.1) \quad |s_m^{(j)}(g)(x)w(x)| \leq C \left(\frac{n}{a_n}\right)^{(2j+1)/2} \left(\int_{A_n} |T(t)^{(2j+1)/4} f(t)w(t)|^2 dt\right)^{1/2}$$

for $|x| \leq a_{2n}$. Hence Lemma 2.2 and the Hölder inequality imply

$$\begin{aligned} \int_{\mathbb{R}} |s_m^{(j)}(g)(x)w(x)|^p dx &\leq 2^p \int_{|x| \leq a_{2n}} |s_m^{(j)}(g)(x)w(x)|^p dx \\ &\leq C \int_{|x| \leq a_{2n}} \left(\frac{n}{a_n}\right)^{p(2j+1)/2} \left(\int_{A_n} |T(t)^{(2j+1)/4} f(t)w(t)|^2 dt\right)^{p/2} dx \\ &\leq C \left(\frac{n}{a_n}\right)^{p(2j+1)/2} \int_{|x| \leq a_{2n}} \left(\int_{|u| \leq \frac{a_{2n}}{2n}} |T(x-u)^{(2j+1)/4} f(x-u)w(x-u)|^2 du\right)^{p/2} dx \\ &\leq C \left(\frac{n}{a_n}\right)^{p(2j+1)/2} a_n^{(2-p)/2} \\ &\quad \times \left\{ \int_{|x| \leq a_{2n}} \left(\int_{|u| \leq \frac{a_n}{n}} |T(x-u)^{(2j+1)/4} f(x-u)w(x-u)|^2 du\right) dx \right\}^{p/2} \\ &\leq C \left(\frac{n}{a_n}\right)^{p(2j+1)/2} a_n^{(2-p)/2} \|T^{(2j+1)/4} fw\|_{L^2(\mathbb{R})}^p \left(\int_{|u| \leq \frac{a_n}{n}} du\right)^{p/2} \\ &\leq C \left(\frac{n}{a_n}\right)^{pj} a_n^{(2-p)/2} \|T^{(2j+1)/4} fw\|_{L^2(\mathbb{R})}^p, \end{aligned}$$

so that we have

$$(6.2) \quad \|v_n^{(j)}(g)w\|_{L^p(\mathbb{R})} \leq C \left(\frac{n}{a_n}\right)^j a_n^{(2-p)/2p} \|T^{(2j+1)/4} fw\|_{L^2(\mathbb{R})}.$$

Next we estimate $v_{n,i}(h)$. Similarly as above, we have

$$\begin{aligned} \int_{\mathbb{R}} |v_{n,i}(h)(x)w(x)|^p dx &\leq 2 \int_{|x| \leq a_{2n}} |v_{n,i}(h)(x)w(x)|^p dx \\ &\leq C \left(\frac{n}{a_n} \right)^{p(2(j-i)-1)/2} a_n^{(2-p)/2} \\ &\quad \times \left\{ \int_{|x| \leq a_{2n}} \left(\int_{B_n \cup C_n} \frac{|T(t)^{(2(j-i)+1)/4} f(t)w(t)|^2}{(x-t)^{2(i+1)}} dt \right) dx \right\}^{p/2}. \end{aligned}$$

Also as in the argument of previous section,

$$\begin{aligned} &\int_{|x| \leq a_{2n}} \left(\int_{B_n} \frac{|T^{(2(j-i)+1)/4}(t) f(t)w(t)|^2}{(x-t)^{2(i+1)}} dt \right) dx \\ &\leq \int_{\mathbb{R}} \left(\int_{\frac{a_n}{n} \leq |u|} \frac{|T^{(2(j-i)+1)/4}(x-u) f(x-u)w(x-u)|^2}{u^{2(i+1)}} du \right) dx \\ &\leq C \left(\frac{n}{a_n} \right)^{2i+1} \|T^{(2(j-i)+1)/4} fw\|_{L^2(\mathbb{R})}^2 \leq C \left(\frac{n}{a_n} \right)^{2i+1} \|T^{(2j+1)/4} fw\|_{L^2(\mathbb{R})}^2 \end{aligned}$$

On the other hand, by (5.6) we have

$$\begin{aligned} &\int_{|x| \leq a_{2n}} \left(T(x)^{(2(j-i)+1)/2} \int_{C_n} \frac{|f(t)w(t)|^2}{(x-t)^{2(i+1)}} dt \right) dx \\ &\leq CT(a_{2n})^{(2(j-i)+1)/2} \int_{\mathbb{R}} \left(\int_{\frac{c_0}{T(a_{2n})} \leq |u|} \frac{|f(x-u)w(x-u)|^2}{u^{2(i+1)}} du \right) dx \\ &\leq C \|fw\|_{L^2(\mathbb{R})}^2 T(a_{2n})^{(2(j-i)+1)/2} \int_{\frac{c_0}{T(a_{2n})} \leq |u|} \frac{1}{u^{2(i+1)}} du \\ &\leq CT(a_{2n})^{(2j+2i+3)/2} \|fw\|_{L^2(\mathbb{R})}^2 \\ &\leq C \left(\frac{n}{a_n} \right)^{2i+1} \|T^{(2j+1)/4} fw\|_{L^2(\mathbb{R})}^2. \end{aligned}$$

Consequently we have

$$(6.3) \quad \|v_{n,i}(h)w\|_{L^p(\mathbb{R})} \leq C \left(\frac{n}{a_n} \right)^j a_n^{(2-p)/2p} \|T^{(2j+1)/4} fw\|_{L^2(\mathbb{R})}$$

for $0 \leq i \leq j$, so that

$$\|v_n^{(j)}(h)w\|_{L^p(\mathbb{R})} \leq C \left(\frac{n}{a_n} \right)^j a_n^{(2-p)/2p} \|T^{(2j+1)/4} fw\|_{L^2(\mathbb{R})}$$

follows. This together with (6.2) shows (1.13). This completes the proof of Theorem 1.3.

Under the same assumptions in Theorem 1.3, the following estimate is also established. Let $\beta > 1$ and $1 \leq p \leq 2$. Then

$$(6.4) \quad \|v_n^{(j)}(f) \frac{w}{(1+|x|)^{(2-p)\beta/(2p)}}\|_{L^p(\mathbb{R})} \leq C \left(\frac{n}{a_n}\right)^j \|T^{(2j+1)/4}fw\|_{L^2(\mathbb{R})}$$

holds for every $T^{(2j+1)/4}fw \in L^2(\mathbb{R})$ and every $n \in \mathbb{N}$.

In fact, in the proof of Theorem 1.3, we used

$$\begin{aligned} & \int_{|x| \leq a_{2n}} \left(\int_{|x-t| \leq \frac{a_{2n}}{2n}} |T(t)^{(2j+1)/4}f(t)w(t)|^2 dt \right)^{p/2} dx \\ & \leq a_n^{(2-p)/2} \left\{ \int_{|x| \leq a_{2n}} \left(\int_{|x-t| \leq \frac{a_n}{n}} |T(t)^{(2j+1)/4}f(t)w(t)|^2 du \right) dx \right\}^{p/2}, \end{aligned}$$

which follows from the Hölder inequality. Instead of this, we use

$$\begin{aligned} & \int_{\mathbb{R}} \frac{1}{(1+|x|)^{(2-p)\beta/2}} \left(\int_{|x-t| \leq \frac{a_{2n}}{2n}} |T(t)^{(2j+1)/4}f(t)w(t)|^2 dt \right)^{p/2} dx \\ & \leq \left(\int_{\mathbb{R}} \frac{1}{(1+|x|)^\beta} dx \right)^{(2-p)/2} \left\{ \int_{\mathbb{R}} \left(\int_{|x-t| \leq \frac{a_n}{n}} |T(t)^{(2j+1)/4}f(t)w(t)|^2 dt \right) dx \right\}^{p/2}. \end{aligned}$$

Then as in (6.2), we obtain

$$\|v_n^{(j)}(g) \frac{w}{(1+|x|)^{(2-p)\beta/(2p)}}\|_{L^p(\mathbb{R})} \leq C \left(\frac{n}{a_n}\right)^j \|T^{(2j+1)/4}fw\|_{L^2(\mathbb{R})}.$$

For the estimate of $v_{n,i}(h)$, we take $w^* \in \mathcal{F}_\lambda(C^3+)$ such that $w^*(x) \sim w(x)/(1+|x|)^{(2-p)\beta/(2p)}$ (see [5, Theorem 4.2]). Then by Lemma 2.2,

$$\int_{\mathbb{R}} \left| v_{n,i}(h) \frac{w(x)}{(1+|x|)^{(2-p)\beta/(2p)}} \right|^p dx \leq 2^p \int_{|x| \leq a_{2n}^*} \left| v_{n,i}(h) \frac{w(x)}{(1+|x|)^{(2-p)\beta/(2p)}} \right|^p dx.$$

By an estimate similar to (6.3), we obtain

$$\|v_{n,i}(h) \frac{w}{(1+|x|)^{(2-p)\beta/(2p)}}\|_{L^p(\mathbb{R})} \leq C \left(\frac{n}{a_n}\right)^j \|T^{(2j+1)/4}fw\|_{L^2(\mathbb{R})},$$

which shows (6.4).

REFERENCES

- [1] H. S. Jung and R. Sakai, Specific examples of exponential weights, Commun. Korean Math. Soc. 24. No.2 (2009), pp.303-319.
- [2] A. L. Levin and D. S. Lubinsky, Orthogonal Polynomials for Exponential Weights, Springer, New York, 2001.
- [3] H. N. Mhaskar, Introduction to the Theory of Weighted Polynomial Approximation, World Scientific, Singapore, 1996.
- [4] R. Sakai and N. Suzuki, Favard-type inequalities for exponential weights, Pioneer J. of Math. Vol 3, No.1 (2011), pp.1-16.

- [5] R. Sakai and N. Suzuki, Mollification of exponential weights and its application to the Markov-Bernstein inequality, Pioneer J. of Math., Vol.7, No.1 (2013), pp.83-101.
- [6] K. Itoh, R. Sakai and N. Suzuki, The de la Vallée Poussin mean and polynomial approximation for exponential weight, arXiv:1311.3337 [math.CA].

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